Fractional Calculus of Schwartz Distributions

by
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Chapter 1

Introduction and historical overview

Fractional calculus refers to integration or differentiation of non-integer order. Interestingly, the field has a history as old as calculus itself. The first recorded discussion of fractional calculus began with Leibniz, where in a letter to L'Hôpital he discussed the differentiation of products functions to differentiation of order 1/2.

However, it was Abel, Liouville and Riemann who layed down the foundations to fractional calculus and these three names dominate the field. In his much celebrated paper on the tautochrone problem [1], Abel was the first to give a physical description of integration of order $1/2$. In fact, the paper went further, and solved the inetgral equation

$$\int_a^x \frac{\varphi(t)}{(x-t)^\beta} dt = f(x) \tag{1.1}$$

for any $0 < \beta < 1$ and $x > a$. The equation (1.1) is called *Abel's integral equation* in his honour. The tautochrone problem itself is the special case with $\alpha = 1/2$, and because of this, many mathematicians erroneously believe Abel only solve the equation when $\beta = 1/2$.

The second step was by Liouville who wrote a series of papers between 1832 and 1837. Here he outlined using an exponential series to differentiate a suitable function. Thus given a function f expressed as a Fourier series:

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x} \tag{1.2}$$

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Figure 1.1: The pioneers of fractional calculus: (from the left) Niels Henrik Abel (1802 - 1829), Joseph Liouville (1809 - 1882) and Georg Friedrich Bernhard Riemann (1826 - 1866).

the differentiation is given by

$$\frac{d^\beta f}{dx^\beta} = \sum_{n=0}^{\infty} c_n a_n^\beta e^{a_n x}. \tag{1.3}$$

Next came Riemann who produced the expression

$$\frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt \tag{1.4}$$

which, as we shall see, is the de facto formula for calculating fractional integration of order α.

Today, fractional integration appears in diverse fields, some in the form of not-so-subtle variation and generalisations. These include solving dual integral equations [11][20] (and the works cited therein), seismic travel times, stereology of spherical particles, spectroscopy of gas discharges, and the refractive index of optical fibers. Books on the classical fractional calculus include Oldham and Spanier [29], Miller and Ross [28], Podlubny [31] and the encyclopedic Samko, Kilbas and Marichev [39], which is now the de facto reference text.

Interestingly, fractional calculus appears to have some unorthodox applications. Traditionally, fractional calculus was studied under the assumptions of μ-analyticity, that is, real functions with convergent series of form $(x-a)^\mu \sum_n a_n(x-$

$a)^n$. However, modern analysis employs concepts from stochastic mechanics, resulting in functions which are continuous everywhere but differentiable nowhere. Historically, these have been considered pathological and ill-suited to applications. For example the Weierstrass function

$$W(x) := \sum_{n=-\infty}^{\infty} a^n e^{i\pi b^n x} \tag{1.5}$$

with $0 < a < 1$, $b > 1$ and $ab \geq 1$ is well known not to have derivatives (Hardy [16]) but it was shown by Bertram Ross, Stefan Samko and Russel Love that it does have fractional derivatives of orders $\alpha < 1$ for $a = b^{-1}$ [35]. In general, if the graph of the function has Hausdorff dimension $2 - s$, then it has fractional derivatives of orders $\alpha < s$ (Kowankar and Gangal [22]). Functions such as (1.5) were labeled "fractal functions" by Rocco and West [34] as their characteristic property is that the graph $(x, f(x))$ of the function is a fractal. They also prove that if a fractal function of dimension D has a fractional derivative of order α, then the resulting function is fractal of dimmension $D + \alpha$. Indeed West et. al. [44] go further, arguing that analyticity is not *relevant* in modern mechanics. The Weierstrass function has been generalized a number of times in the literature, the most interesting being the generalized Weierstrass function (GWF) defined as

$$W(t) := \sum_{n=-\infty}^{\infty} \frac{(1 - e^{i\gamma^n t})}{\gamma^{(2-D)n}} e^{i\varphi_n} \tag{1.6}$$

with $1 < D < 2$, $\gamma > 1$ and φ_n a phase (which is allowed to be either deterministic or stochastic). The GWF, under these parameters is continuous but not differentiable, has no characteristic scale, and has dimension D [3][26]. For a discussion of the fractional derivatives of the GWF, see Rocco and West [34].

Figure 1.2: Karl Theodor Wilhelm Weierstrass (1815 - 1897)

In his paper [9] Paul Dirac introduced the "delta function" defined as

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \tag{1.7}$$

which is well-defined function from $\mathbb{R} \to \mathbb{R} \cup \{\infty\}$. But, at first glance the definition of the integral is not defined. Dirac defined it as

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a) \tag{1.8}$$

which implies that $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$. However, from the theory of Lebesgue integration, because the delta function is zero almost everywhere, it should also have zero integral. In other words, the delta function breaks integration.

This spurred mathematicians into trying to justify this mathematical blasphemy. The pioneer was Siegi Sobolev, who considered the classical definition of the derivative to be too restrictive. Using the well known concept of integration by parts, he introduced the concept of what today are called *weak derivatives* (Sobolev himself called these "generalized derivatives"). Here, the weak derivative of f is defined as a function g such that

$$\int_{-\infty}^{\infty} g(x)\varphi(x)\, dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\, dx \tag{1.9}$$

for some suitably chosen φ.

This concept was extended by Laurent Schwartz, who considered objects *not necessarily functions* which still satisfied (1.9). Thus, for Schwartz, the derivative of function may not be a function, but objects more general than functions, what he called *distributions*. Since his monograph *Théorie des Distributions* [41] distributions have been extensively studied. Today, more powerful concepts such as Sato hyperfunctions [40] and Colombeau generalized algebras [8] have been constructed and are used extensively.

In Chapter 2 we will give an overview of the Riemann-Liouville fractional integral and derivitaive and the Caputo fractional derivative. We will study its general properties and give a visual interpretation of these operators. In Chapter 3, we will give an overview of Schwartz distributions, including tempered distributions which have prime importance in physics. Chapter 4 looks at two extensions of

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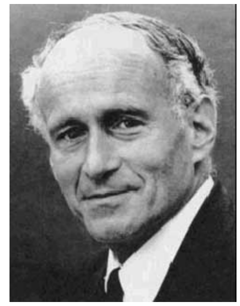

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Figure 1.3: The pioneers of generalized functions: (from the left) Paul Adrien Maurice Dirac (1815 - 1897), Sergei Lvovich Sobolev (1908 - 1989) and Laurent Schwartz (1915 - 2002).

factional calculus to distributions. In Chapter 5 we give a short overview of some applications of fractional calculus. In the introduction it was mentioned that there are some not-so-subtle generalizations of fractional calculus. We will mention two such operators in the Appendix.

Chapter 2

Fractional Calculus

The term *fractional* is a misnomer, as fractional calculus refers to any non-integer order integro-differentiation, whether that be rational, irrational or complex. This section concentrates on the generalization of the integral discovered by Riemann and Liouville (hence the name). It is the simplest example of fractional calculus (others do exist) and also probably the most important in one dimension. The simplest properties are given and proved. The Fourier transform of fractional integro-differentitation is given, and in fact can be used as the definition itself. For a student who is less interested in rigorous derivation and properties, Section 2.3 considers a graphical interpretation of fractional integration.

2.1 Riemann-Liouville differintegral

Consider integrating a function twice over the same interval (a, x)

$$\int_a^x \int_a^{x_1} \varphi(t)\,dt\,dx_1.$$

Then we can use integration by parts to obtain

$$\int_a^x \int_a^{x_1} \varphi(t)\,dt\,dx_1 = \int_a^x (x-t)\varphi(t)\,dt. \tag{2.1}$$

Indeed, we let $u = x - t, dv = \varphi(t)\,dt$ so then $du = -dt$ and $v = \int_a^t \varphi(t')\,dt'$. Then the right hand side equals

$$\int_a^x (x-t)\varphi(t)\,dt = \left[(x-t)\int_a^t \varphi(t')\,dt'\right]_{t=a}^{t=x} + \int_a^x \int_a^{x_1} \varphi(t)\,dt\,dx_1$$

which equals the left hand side. Similarly, one can prove the Cauchy formula for repeated integration

$$\int_a^x \int_a^{x_{n-1}} \dots \int_a^{x_1} \varphi(t)\,dt\,dx_1 \dots dx_{n-1} = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)\,dt. \tag{2.2}$$

To prove this we introduce the notation $\varphi^{[k]}(x) = \int_a^x \varphi^{[k-1]}(t)\,dt$, $\varphi^{[0]}(x) = \varphi(x)$, to represent the k-th antiderivative. Then letting $u = (x-t)^{n-1}, dv = \varphi(t)\,dt$ so that $du = -(n-1)(x-t)^{n-2}\,dt, v = \varphi^{[1]}(t)$. Then

$$\int_a^x (x-t)^{n-1}\,\varphi(t)\,dt = (n-1)\int_a^x (x-t)^{n-2}\varphi^{[1]}(t)\,dt.$$

We need to repeat this procedure until the $(x-t)$ term vanishes. After k steps we have

$$(n-1)(n-2)\dots(n-k)\int_a^x (x-t)^{n-1-k}\varphi^{[k]}(t)\,dt$$

and so after $n-1$ steps

$$(n-1)(n-2)\dots(2)(1)\int_a^x \varphi^{[n-1]}(t)\,dt = (n-1)!\,\varphi^{[n]}(t)$$

which is n-fold integration.

Because the gamma function directly generalizes the factorial $\Gamma(n) = (n-1)!$, the right hand side of the equation can be generalized to any order that the gamma function is defined for. We take this to be our definition.

Definition 2.1.1. Let φ be an absolutely continuous function over (a, b). Then the left and right Riemann-Liouville fractional integrals (respectively) are

$$(I^\alpha \varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \varphi(u)(x-u)^{\alpha-1}du,\ x > a. \tag{2.3}$$

$$(K^{\alpha}\varphi)(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \varphi(u)(x-u)^{\alpha-1} du, \; x < b, \tag{2.4}$$

Further we define the fractional derivative using post-differentiation

$$D^{\alpha}\varphi(x) := \frac{d}{dx} I^{1-\alpha}\varphi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{a}^{x} \frac{\varphi(u)}{(x-u)^{\alpha}} \, du. \tag{2.5}$$

There are some immediate consequences of this definition. The most important is that the base point *a must* be specified. Because we have defined fractional differentiation through integration, fractional derivatives are no longer local operations. They are defined over an interval. This may be why Leibinz believed it was a paradox: he wished the derivative to be unique and local.

Secondly, because the Gamma function is defined for all $\mathbb{C} \setminus \{0, -1, \ldots\}$, we can define fractional calculus of *complex* order, however, in this case the power function requires a branch cut (usually chosen to be a ray originating from a and passing through the origin). However, in this paper, we will only consider real orders.

Thirdly, the space of functions chosen above are the functions which are absolutely continuous. This is because absolute continuity is known to be a sufficient condition to fractional integration. Nonetheless other function spaces have been studied in the literature, including the fractal functions discussed in the Introduction, and the concepts of α-continuity and α-integrability (Bonilla et al. [2]). We will not concern ourselves with these spaces, and later, will replace absolute continuity the concept of μ-analyticity, which is more natural in applications. Similarly, there is ambiguity in whether we are specifying the Riemann integral, or the Lebesgue integral. Here, by default, we will assume the latter.

Different textbooks use different notation, the left and right integrals being written ${}_aI_x^{\alpha}$ and ${}_xI_b^{\alpha}$ in Podlubny [32], while they are written as I_{a+}^{α} and I_{b-}^{α} in Samko et al. [39]. The terminals are dropped in this paper as we will specify them explicitly in the text if the case arises. The use of K to denote the right integration is due to McBride [27], in honour of Hermann Kober.

Fractional integrals satisfy what is known as the semigroup property: $I^{\alpha}I^{\beta}\varphi(x) = I^{\alpha+\beta}\varphi(x)$ for $\alpha, \beta > 0$. This is true in any point if $\varphi \in C([a, b])$. Indeed direct evaluation gives

$$I^{\alpha}I^{\beta}\varphi(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{a}^{x} (x-t)^{\alpha-1} \int_{a}^{t} (t-\tau)^{\beta-1} \, \varphi(\tau) \, d\tau \, dt$$

whence applying Fubini's theorem

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_\tau^x (x-t)^{\alpha-1}(t-\tau)^{\beta-1}\varphi(\tau)\,dt\,d\tau$$

and then letting $t = \tau + s(x-\tau)$ we have $dt/ds = x - \tau$ so that $(x-t)^{\alpha-1}(t-\tau)^{\beta-1}$ becomes $(x-\tau)^{\alpha+\beta-1}(x-\tau)^{-1}s^{\beta-1}(1-s)^{\alpha-1}$ and we obtain

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_0^1 (x-\tau)^{\alpha+\beta-1}s^{\beta-1}(1-s)^{\alpha-1}\varphi(\tau)\,ds\,d\tau.$$

Using the definition of the beta function

$$B(p,q) := \int_0^1 s^{p-1}(1-s)^{q-1}\,ds = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \text{ for } p,q > 0. \tag{2.6}$$

we obtain

$$\begin{aligned} I^\alpha I^\beta\varphi(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x (x-\tau)^{\alpha+\beta-1}\varphi(\tau)\,d\tau\int_0^1 (x-\tau)^{\alpha+\beta-1}\,ds \\ &= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x (x-\tau)^{\alpha+\beta-1}\,\varphi(\tau)\,d\tau. \end{aligned} \tag{2.7}$$

Fractional derivatives in general however do not commute, which is a major difference with integer order derivatives. The following theorem proved in Samko et al. [39] gives sufficient conditions for the semigroup property:

Theorem 2.1.2. *The semigroup property*

$$I^\alpha I^\beta\varphi(x) = I^{\alpha+\beta}\varphi(x)$$

with $\alpha,\beta \neq 0$ *is valid provided one of the following hold:*

1. $\beta > 0, \alpha+\beta > 0, \varphi \in L_1(a,b)$
2. $\beta < 0, \alpha > 0, \varphi \in D^\beta(L_1)$

3. $\alpha < 0, \alpha + \beta < 0, \varphi \in D^{\alpha+\beta}(L_1)$.

As for K^α, that is proved by its relationship with I^α through the reflection operator $(Q\varphi)(x) := \varphi(a + b - x)$,

$$QI^\alpha = K^\alpha Q, \quad QK^\alpha = I^\alpha Q. \tag{2.8}$$

From this we obtain the fractional integration by parts formula:

$$\int_a^b \varphi(x)\,(I^\alpha\,\psi)(x)\,dx = \int_a^b (K^\alpha\varphi)(x)\,\psi(x)\,dx \tag{2.9}$$

where $\varphi \in L^p(a,b), \psi \in L^q(a,b)$ and $1/p + 1/q = 1$ first proved by Love [24].

2.2 Calculating fractional integrals

To obtain some intuition, let us start by calculating a few examples from Butzer and Westphal [5].

Example 2.2.1. Let the base point be the origin, and for $b > -1$, consider the power function $f(x) = x^b$. Then the fractional derivative of order α is equal to

$$D^\alpha x^b = \frac{\Gamma(b+1)}{\Gamma(b-\alpha+1)} x^{b-\alpha}.$$

Let $b > -1$, $a = 0$ and consider the function $f(x) = x^b$. We will use the definition of the beta function (2.6).

$$\begin{aligned}
D^\alpha x^b &= \frac{d^m}{dx^m}\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-u)^{m-\alpha-1}u^b\,du \\
&= \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\left[x^{m-\alpha+b}\int_0^1 (1-v)^{m-\alpha-1}v^b\,dv\right] \\
&= \frac{1}{\Gamma(m-\alpha)}\frac{\Gamma(b+1)\Gamma(m-\alpha)}{\Gamma(b+1+m-\alpha)}\frac{d^m}{dx^m}x^{m-\alpha+b} \\
&= \frac{\Gamma(b+1)}{\Gamma(b+1+m-\alpha)}x^{b-\alpha}\frac{\Gamma(m-\alpha+b+1)}{\Gamma(m-\alpha+b-m+1)} \\
&= \frac{\Gamma(b+1)}{\Gamma(b-\alpha+1)}x^{b-\alpha}
\end{aligned} \tag{2.10}$$

where we made use of, with $p \in \mathbb{R} \setminus \{-1, -2, \ldots\}$

$$\frac{d^m}{dx^m} x^p = \frac{\Gamma(p+1)}{\Gamma(p-m+1)} x^{p-m} \tag{2.11}$$

If we substitute $b = 0$ then we obtain $D^\alpha c = x^{-\alpha}/\Gamma(1-\alpha)$ for any $\alpha > 0$. This leads to the counter-intuitive result where the derivative of the constant is non-zero for non-integer α and moreover, it is unbounded at $x = 0$. We will return to this problem later. Further, observe that if $b = \alpha - 1$ we obtain $D^\alpha x^{\alpha-1} = \Gamma(\alpha)/x\Gamma(0)$. When one uses the convention $1/\Gamma(0) = 0$, this is analogous to the classical result $D^2 x = 0$. When one considers indefinite integrals, one obtains undetermined constants. Similarly, if one were to consider an "indefinite fractional integral" there would be a "constant term" of $Cx^{\alpha-1}$.

Example 2.2.2. Consider the exponential function with $a = -\infty$. We have

$$I^\alpha e^{px} = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-u)^{\alpha-1} e^{pu} du \tag{2.12}$$

To evaluate this, we use $x - u = y/p$ so that $dy/du = -p$ to get

$$\begin{aligned} I^\alpha e^{px} &= \frac{1}{\Gamma(\alpha)} \int_{\infty}^{0} \left(\frac{y}{p}\right)^{\alpha-1} e^{px-y} \frac{dy}{-p} \\ &= \frac{1}{\Gamma(\alpha)} \frac{e^{px}}{p^{\alpha-1}} \frac{1}{p} \int_{0}^{\infty} y^{\alpha-1} e^{-y} dy \\ &= \frac{e^{px}}{p^\alpha} \end{aligned} \tag{2.13}$$

A word of caution. It is only because the lower terminal is $-\infty$ that the fractional integral of the exponential, is an exponential. Later we will see that if the lower terminal is 0, we will obtain a different function.

2.3 Geometric interpretation of the fractional integral

The above section is well and good, however, many students ask what fractional calculus *looks* like. This is probably due to the differentiation being taught as the

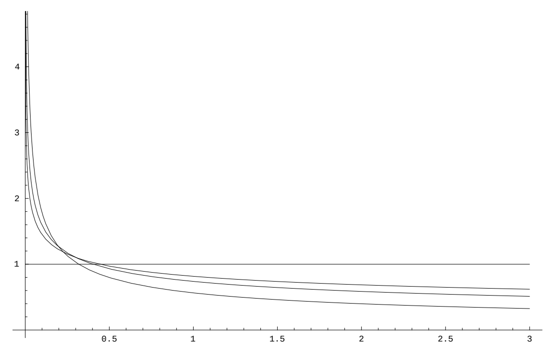

Figure 2.1: Riemann-Liouville fractional derivatives of $f(x) = 1$ for $\alpha = 0, 1/4, 1/3, 1/2$. Observe that the derivatives converge to $f(x) = 0$ except for an asymptote at $x = 0$.

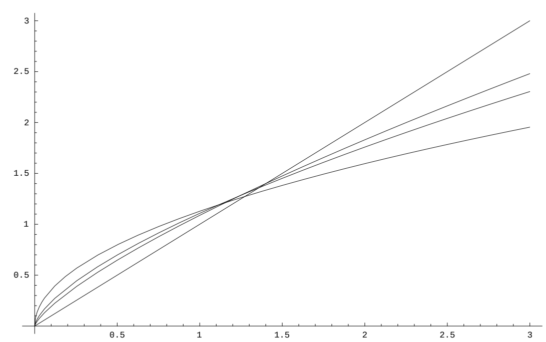

Figure 2.2: Riemann-Liouville fractional derivatives of $f(x) = x$ for $\alpha = 0, 1/2, 1/3, 1/4$. Observe how the derivatives approach $f(x) = 1$, but each derivative passes through the origin.

gradient of the tangent, and the integral as the area under the curve (as opposed to how the integral is constructed on an abstract measure space). This section looks at how to interpret the formulae (2.3) and (2.4).

Let us follow Podlubny [32] and fix our base point $a = 0$. Then I^α can be represented as a Lebesgue-Stieltjes integral by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau = \int_0^t f(\tau)\, dg_t \tag{2.14}$$

where

$$g_t(\tau) = \frac{t^\alpha - (t-\tau)^\alpha}{\Gamma(\alpha+1)}. \tag{2.15}$$

Let us take axes τ, g and f. In the (τ, g) we plot $g_t(\tau)$. Then we build what Bullock [4] calls a "fence", we plot the function $f(\tau)$ along the plot of g_t we just created. So the top of the fence is a three-dimensional curve $(\tau, g_t(\tau), f(\tau))$ for $0 \leq \tau \leq t$.

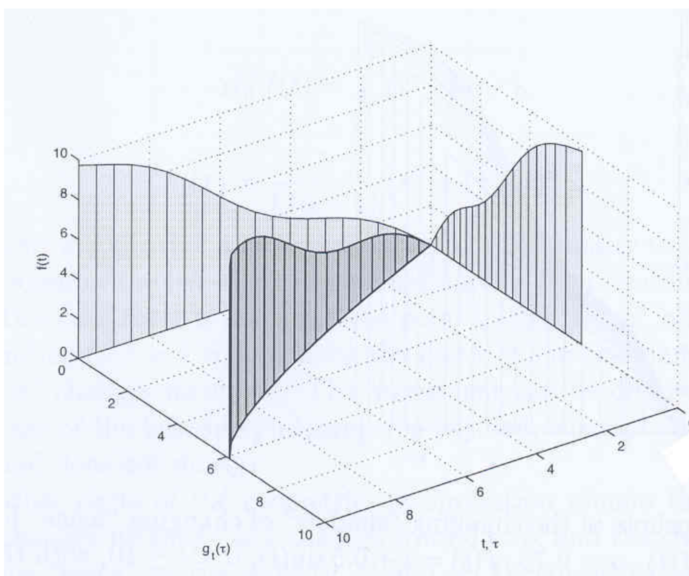

Figure 2.3: The fence and the shadows, from Podlubny [32]. The shadow on the left represents the standard integral, the shadow on the right represents the fractional integral.

The fence can be projected onto either the (τ, f) plane or the (g, f) plane. The former case corresponds to the the standard integral, that is, the area of the

projection ("shadow") is the value of

$$\int_0^t f(\tau)\, d\tau;$$

while the latter corresponds to (2.14). When $\alpha = 1$ then g_t is the identity, and so both shadows are equal. Thus geometrically, the Riemann-Liouville fractional integral generalizes the standard notion of are under the curve.

For fractional derivatives the difficulty stems from the desire to have fractional derivatives of functions with no tangent. A possible solution was given in Carpenteri et al. [7]. It is based on the observation that if a function f has fractional derivatives up to some maximal $\alpha > 0$, then from [22], it can be shown there is local Taylor expansion

$$f(x) = f(x_0) + \frac{D_a^\alpha f(x_0)}{\Gamma(\alpha + 1)}(x - x_0)^\alpha + R(x) \tag{2.16}$$

where $R(x)$ is a negligible remainder function and D_a^α is the *local fractional derivative* defined as

$$D_a^\alpha f(x_0) = \lim_{x \to x_0} \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_{x_0}^{x} \frac{f(u) - f(x_0)}{(x - u)^{\alpha - 1}}\, du. \tag{2.17}$$

Setting $\alpha = 1$ in (2.16) gives the standard Taylor polynomial. Thus, near the point x_0, $f(x)$ behaves similarly to the power function $(x - x_0)^\alpha$. Carpenteri et al. [7] argue that this the generalization of the concept of "tangent of a curve" which is required.

2.4 Analytic functions and Leibniz rule

Here we consider ourselves with the Leibniz Rule

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(n)} \tag{2.18}$$

generalized to non-integer orders. There are a number of proofs of this result, for example, Osler [30] uses contour integration. We will prove this result for a series. This means the rule only holds for μ-analytic functions, but for most applications that is sufficient.

Lemma 2.4.1. *Suppose $f(x) = \sum_{n\geq 0} f_n(x)$ where $f_n \in C([a,b])$ is uniformly convergent. Then term-wise integration is possible,*

$$\left(I^\alpha \sum_{n=0}^{\infty} f_n\right)(x) = \sum_{n=0}^{\infty} \left(I^\alpha f_n\right)(x). \tag{2.19}$$

Proof. We wish to evaluate

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} \sum_{n=0}^{\infty} f_n(u)\, du. \tag{2.20}$$

Now because $(x-u)^{\alpha-1}$ does not depend on n, it can be taken inside the sum. But because $\sum f_n$ converges uniformly on $[0,x]$, $\sum f_n(u)(x-u)^{\alpha-1}$ is also a uniformly convergent series. The integral of a uniformly convergent series is the sum of the integrals thus

$$\begin{aligned} I^\alpha\left(\sum_n f_n\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} \sum_{n=0}^{\infty} f_n(u)\, du \\ &= \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \int_0^x f_n(u)(x-u)^{\alpha-1}\, du \\ &= \sum_{n=0}^{\infty} \left(I^\alpha f_n\right)(x) \end{aligned}$$

as required. Existence stems from the fact that a uniformly convergent series of continuous functions is absolutely continuous on a compact set. □

Lemma 2.4.2. *Let $D^\alpha f_n$ exist for each n, and suppose that for every subinterval $[a+\epsilon, b]$ the series $\sum f_n$ and $\sum D^\alpha f_n$ are uniformly convergent. Then*

$$\left(D^\alpha \sum_{n=0}^{\infty} f_n\right)(x) = \sum_{n=0}^{\infty} \left(D^\alpha f_n\right)(x). \tag{2.21}$$

Proof. Suppose that $0 < \alpha < 1$ then

$$D^\alpha f = \frac{d}{dx} I^{1-\alpha} f.$$

Applying Lemma 2.4.1 we obtain

$$\frac{d}{dx}\sum_{n=0}^{\infty}\left(I^{\alpha} f_n\right)(x).$$

Now because $\sum D^{\alpha} f_n = \sum d/dx\,(I^{1-\alpha} f_n)$ is uniformly convergent on every compact subinterval, we can differentiate term-by-term and obtain the desired result. As for $\alpha > 1$ we need to find $m \in \mathbb{N}$ such that $m \leq \alpha < m+1$ and then repeat the argument on

$$D^{\alpha} f = \frac{d^{m+1}}{dx^{m+1}} I^{1-\alpha-m} f.$$

□

Lemma 2.4.3. *Let $f(x)$ be analytic on (a,b) then*

$$(D^{\alpha} f)(x) = \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{f^{(n)}(x)}{\Gamma(n+1-\alpha)}(x-a)^{n-\alpha}. \tag{2.22}$$

Proof. We start with integration. Let $\alpha < 0$ and so

$$(D^{\alpha} f)(x) = \frac{1}{\Gamma(-\alpha)}\int_{a}^{x}(x-t)^{-\alpha-1} f(t)dt. \tag{2.23}$$

Since $f(t)$ is analytic we can expand around x

$$f(t) = \sum_{n=0}^{\infty}\frac{(-1)^n f^{(n)}(x)}{n!}(x-t)^n. \tag{2.24}$$

We apply (2.4.1) to (2.24) to obtain the required expression. The case $\alpha = 0$ reduces to triviality. Suppose now that $\alpha > 0$ and fix an integer $m \leq \alpha < m$, we have from 2.4.2

$$\begin{aligned}
(D^{\alpha} f)(x) &= \frac{d^{m+1}}{dx^{m+1}}(D^{\alpha-m-1}) f(x) \\
&= \left(\frac{d}{dx}\right)^{m+1}\sum_{n=0}^{\infty}\binom{\alpha-m-1}{n}\frac{f^{(n)}(x)}{\Gamma(2-\alpha+m+n)}(x-a)^{n-\alpha+m+1} \\
&= \sum_{n=0}^{\infty}\binom{\alpha-m-1}{n}\left(\frac{d}{dx}\right)^{m+1}\frac{f^{(n)}(x)}{\Gamma(2-\alpha+m+n)}(x-a)^{n-\alpha+m+1} \qquad (2.25)
\end{aligned}$$

by applying (2.4.2) (which is possible as a Taylor series is uniformly convergent)

$$(D^\alpha f)(x) = \sum_{n=0}^{\infty} \binom{\alpha - m - 1}{n} \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{f^{(n+k)}(x)(x-a)^{n-\alpha+k}}{\Gamma(n-\alpha+k+1)} \tag{2.26}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{\alpha - m - 1}{n} \binom{m+1}{k} \frac{f^{(n+k)}(x)(x-a)^{n-\alpha+k}}{\Gamma(n-\alpha+k+1)}. \tag{2.27}$$

we are allowed to increase the term of summation as the binomial coefficents are zero for higher terms. Letting $j = n + k$ we have

$$(D^\alpha f)(x) = \sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} \binom{\alpha - m - 1}{n} \binom{m+1}{j-n} \right) \frac{f^{(j)}(x)}{\Gamma(j-\alpha+1)} (x-a)^{j-\alpha}. \tag{2.28}$$

But by the identity

$$\sum_{j=0}^{k} \binom{\alpha}{j} \binom{\beta}{k-j} = \binom{\alpha+\beta}{k}$$

we have (2.22). □

Corollary 2.4.4. *If a function has expression of the form* $f(x) = (x-a)^\mu \sum_{n=0}^{\infty} c_n (x-a)^n$ *in a neibourhood of a then it has fractional derivative of form*

$$D^\alpha f(x) = (x-a)^{\mu-\alpha} \sum_{n=0}^{\infty} \frac{c_n \Gamma(n+\mu+1)}{\Gamma(n-\alpha+\mu+1)} (x-a)^n. \tag{2.29}$$

Proof. Because power series are uniformly convergent, from Lemma 2.4.3 we can apply term wise fractional differentiation to the series $\sum_n c_n (x-a)^{n+\mu}$ using

$$D^\alpha (x-a)^b = \frac{\Gamma(\beta+1)}{\Gamma(b-\alpha+1)} (x-a)^{b-\alpha} \tag{2.30}$$

to obtain (2.29). □

The above result shows that μ-analyticity is a sufficient condition for a function to have fractional derivatives. This is significant because many physical problems are modeled by these functions. Finally, we give the generalized Leibniz rule:

Theorem 2.4.5. *Let* f, g *be analytic on* $[a, b]$*. Then*

$$D^\alpha (fg)(x) = \sum_{n=0}^{\infty} (D^n f)(x)\, g^{(\alpha-n)}(x). \tag{2.31}$$

Proof. From (2.4.3) we have (because fg is analytic)

$$D^{\alpha}(fg)(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(fg)^{(n)}(a)}{\Gamma(n+1-\alpha)}(x-a)^{n-\alpha}$$

and then applying the standard Leibniz rule, and then intechanging the order of summation we have

$$\begin{aligned}
D^{\alpha}(fg) &= \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{j=0}^{\infty} \binom{k}{j} f^{(k-j)} g^{j} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \binom{\alpha}{k}\binom{k}{j} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k-j)} g^{j} \\
&= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \binom{\alpha}{m+j}\binom{m+j}{j} \frac{(x-a)^{m+j-\alpha}}{\Gamma(m+j-\alpha+1)} f^{(m)} g^{(j)} \qquad (2.32)
\end{aligned}$$

with $m = k - j$. We use the identity $\binom{\alpha}{m+j}\binom{m+j}{j} = \binom{\alpha}{m}\binom{\alpha-m}{j}$ to obtain

$$\begin{aligned}
D^{\alpha}(fg) &= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \binom{\alpha}{m}\binom{\alpha-m}{j} \frac{(x-a)^{m+j-\alpha}}{\Gamma(m+j-\alpha+1} D^m f \, D^j g \\
&= \sum_{n=0}^{\infty} \binom{\alpha}{m} D^m f \sum_{j=0}^{\infty} \binom{\alpha-m}{j} D^j g \frac{(x-a)^{m+j-\alpha}}{\Gamma(m+j-\alpha+1} \\
&= \sum_{m=0}^{\infty} \binom{\alpha}{m} f^{(m)} g^{(\alpha-m)} \qquad (2.33)
\end{aligned}$$

as required. □

Example 2.4.6. Consider the exponential $e^x = \sum_n x^n/n!$. Then the fractional derivative, with base point at $a = 0$ is given by Lemma 2.4.3

$$D^{\alpha} e^x = x^{-\alpha} E_{1,1-\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^{n-\alpha}}{\Gamma(n+1-\alpha)}. \qquad (2.34)$$

The function $E_{\alpha,\beta}$ is called the *two-parameter Mittag-Leffler function*. Observe that this is different to to the case when $a = -\infty$ (Example 2.2.2). The fractional derivatives of e^x are shown in Figure 2.4.

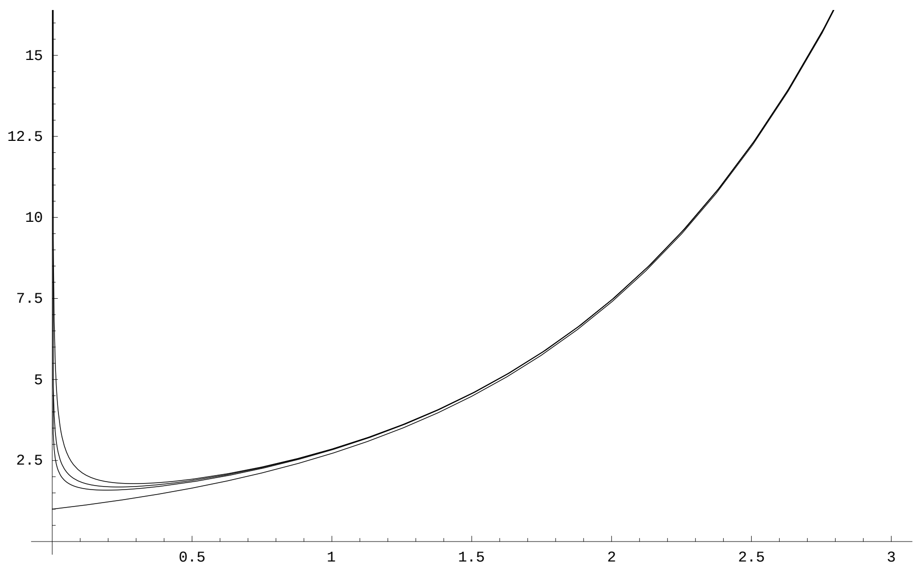

Figure 2.4: Riemann-Liouville fractional derivatives of $f(x) = e^x$ with base point at $a = 0$ for $\alpha = 0, 1/2, 1/3, 1/4$. Observe how at small x the $x^{-\alpha}$ term dominates the Mittag-Leffler function.

2.5 Fractional integration and integral transforms

Observe that the definition of Riemann-Liouville integration is similar to Fourier convolution when $a = -\infty$ and Laplace transform when $a = 0$. Thus it is not unreasonable to expect that these play a major role in fractional calculus.

Fourier transform of the Riemann-Liouville fractional integral

The Fourier transform is defined here as

$$\mathcal{F} f(x) = \hat{f}(x) := \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt \tag{2.35}$$

and so it should not be a surprise that Fourier transform techniques are used in applications when the lower limit $a = -\infty$. First recall that

$$\int_{0}^{\infty} t^{\alpha-1} e^{-zt}\, dt = \frac{\Gamma(\alpha)}{z^{\alpha}} \tag{2.36}$$

for real α and $z \neq 0$, by the definition of the Gamma function. Secondly, recall that the Fourier transform can be written as derivative:

$$\mathcal{F}\varphi(x) = \frac{d}{dx}\int_{-\infty}^{\infty} \frac{e^{ixt}-1}{it}\,\varphi(t)\,dt. \tag{2.37}$$

Theorem 2.5.1. *Let $0 < \alpha < 1$ and φ Lebesgue integrable, then the Fourier transform of the Riemann-Liouville integral is given by*

$$\mathcal{F}I^{\alpha}\varphi(x) = \frac{\hat{\varphi}(x)}{(\mp ix)^{\alpha}} \tag{2.38}$$

where the Fourier transform is understood in terms of (2.36) and

$$(\mp ix)^{\alpha} = |x|^{\alpha}\, e^{\mp \operatorname{sign}(x)\,\alpha\pi i/2}.$$

Proof. We apply Fubini's Theorem to

$$\begin{aligned}
\mathcal{F}I^{\alpha}\varphi(x) &= \frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{e^{ixt}-1}{it}\int_{-\infty}^{t}\frac{\varphi(s)\,ds}{(t-s)^{1-\alpha}}dt \\
&= \frac{1}{\Gamma(\alpha)}\frac{d}{dx}\int_{-\infty}^{\infty}\varphi(s)\int_{s}^{\infty}\frac{e^{ixt}-1}{it(t-s)^{1-\alpha}}\,dt\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\infty}\varphi(s)\,e^{ixs}\int_{0}^{\infty}\frac{e^{ix\tau}}{\tau^{1-\alpha}}\,d\tau\,ds \\
&= \frac{\hat{\varphi}(x)}{\Gamma(\alpha)}\int_{0}^{\infty}\frac{e^{ixt}\,dt}{t^{1-\alpha}} = \frac{\hat{\varphi}(x)}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{(-ix)^{\alpha}}
\end{aligned}$$

with $z = it$ and the result follows. □

For a discussion of the cosine and sine transforms, see Samko et al. [39].

Laplace transform of the Riemann-Liouville fractional integral

We will assume that the lower terminal a is equal to zero. We let $\varphi_{\alpha}(x) = x^{\alpha-1}$ and then we can write the fractional integral as a Laplace convolution

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(u)(x-u)^{\alpha-1} du = \varphi_\alpha(x) * \frac{f(x)}{\Gamma(\alpha)}. \tag{2.39}$$

Since the Laplace transform of φ_α is $\tilde{\varphi}_\alpha(s) = s^\alpha \Gamma(\alpha)$, we can write

$$\mathcal{L} I^\alpha f(x) = s^\alpha \tilde{f}(s). \tag{2.40}$$

As for the derivative we use standard properties for the Laplace transform

$$\mathcal{L} D^n f(x) = s^n \tilde{f}(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0+), \quad n \in \mathbb{N}$$

and so because $D^\alpha = d/dx\, I^{1-\alpha}$ we have

$$\mathcal{L} D^\alpha f(x) = s^n \mathcal{L}\{I^{n-\alpha} f\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-k-1}}{dx^{n-k-1}} I^{n-\alpha} f(0+) \tag{2.41}$$

$$= s^\alpha \tilde{f}(s) - \sum_{k=0}^{n-1} s^k \, (D^{\alpha-k-1} f)(0+). \tag{2.42}$$

2.6 The Caputo fractional derivative

The Riemann-Liouville fractional derivative has a number of problems. Firstly, the fractional derivative of a constant is non-zero. Second, and more importantly, if one takes the Lapace transform, one needs to calculate expressions of the form

$$\lim_{x \to 0} D^{\alpha-1} f(x) \tag{2.43}$$

which have next to no physical meaning. In lieu of this Caputo [6] proposed a different definition of the fractional derivative. We follow Podlubny [31] in notation.

Definition 2.6.1. Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $n - 1 < \alpha < n$. Then the *Caputo fractional derivative* of order α is

$${}^C D^\alpha f(x) := \frac{1}{\Gamma(\alpha - n)} \int_a^x \frac{f^{(n)}(u)\, du}{(x-u)^{\alpha-n+1}}, \quad (n - 1 < \alpha < n)$$

There are some immediate consequences. Firstly, the fractional derivative of a constant is always zero. Secondly, as $\alpha \to n$ we obtain

$$\lim_{\alpha \to n} {}^{C}D^{\alpha} f(x) = \lim_{\alpha \to n} \frac{f^{(n)}(a)(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} \tag{2.44}$$

$$+ \frac{1}{\Gamma(n-\alpha+1)} \int_a^x (x-u)^{n-\alpha} f^{(n-\alpha)}(u)\, du \tag{2.45}$$

$$= f^{(n)}(a) + \int_a^x f^{(n+1)}(u) du = f^{(n)}(x). \tag{2.46}$$

Thirdly the Laplace transform gives

$$\mathcal{L}^{C} D^{\alpha} f(x) = s\tilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). \tag{2.47}$$

That is, we can use the initial values which represent physical quantities. Indeed, let f be a function which has a Laplace transform for each derivative, and $p > 0$ with $p = n + \alpha$ (n is a natural number). Then we have

$$\int_0^\infty \left(\frac{d^{n+\alpha}}{dt^{n+\alpha}} f(t) \right) e^{-st} dt = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \left\{ \int_0^t (t-\xi)^{-z} f^{(n+1)}(\xi)\, d\xi \right\} e^{-st} dt. \tag{2.48}$$

Now we wish to change the order of integration, which is possible because

$$|f^{(n+1)}| \int_\xi^\infty (t-\xi)^{-p} e^{-st} dt = |f^{(n+1)}| e^{-sq} s^{p-1} \Gamma(1-z) \tag{2.49}$$

and this is integrable on $(0, \infty)$ because $f^{(n+1)}$ has a Laplace transform by hypothesis. So we interchange the order of (2.48) to obtain and use integration by parts

to obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)}\int_0^\infty \left\{\int_0^t (t-\xi)^{-z} f^{(n+1)}(\xi)d\xi\right\} e^{-st}dt \\
&\quad = \frac{1}{\Gamma(1-z)}\left[f^{(n)}(\xi)\int_\xi^\infty (t-\xi)^{-z}e^{st}dt\right]_0^\infty \\
&\quad - \int_0^\infty f^{(m)}(\xi)\frac{d}{d\xi}\left[\frac{1}{\Gamma(1-z)}\int_\xi^\infty (t-\xi)^{-z}e^{-st}\right]d\xi \\
&\quad = s^{z-1}f^{(n)}(0) - \int_0^\infty f^{(n)}(\xi)\frac{d}{d\xi}\left[\frac{1}{\Gamma(1-z)}\int_\xi^\infty (t-e)^{-z}e^{-st}dt\right]d\xi \\
&\quad = s^{z-1}f^{(n)}(0) - \int_0^\infty \frac{1}{\Gamma(1-z)}f^{(n)}(\xi)\frac{d}{d\xi}\left\{\left[\frac{(t-\xi)^{1-z}e^{-st}}{1-z}\right]_0^\infty\right. \\
&\quad \left. + \frac{s}{1-z}\int_\xi^\infty (t-\xi)^{1-z}e^{-st}\right\}d\xi \\
&\quad = s^{z-1}f^{(0)} + \int_0^\infty \frac{s}{\Gamma(1-z)}f^{(n)}(\xi)\left[\int_\xi^\infty (t-\xi)^{-z}e^{-st}dt\right]d\xi \\
&\quad = s^{z-1}f^{(n)}(0) + s^z\int_0^\infty f^{(m)}(\xi)e^{-s\xi}d\xi
\end{aligned}
$$

and the result follows by induction. This result, and the fact that the fractional derivative of constants is zero for all orders $\alpha > 0$, is why the Caputo fractional derivative appears to be more favored in the physical sciences. Nonetheless, unlike in Riemann-Liouville fractional derivatives, we require the function f to have derivatives of order n, which is quite a large restriction. Interestingly, if $f(0) = 0$, then the Caputo fractional derivative is equal to the Riemann-Liouville fractional derivative. See Podulbny [31] for details.

Recently, the Caputo fractional derivative was generalized by Hilfer [18] [17]:

Definition 2.6.2. The Hilfer left fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \mu \leq 1$ with respect to x is defined by

$$D^{\alpha,\mu} f(x) := \left(I^{\mu(1-\alpha)} \frac{d}{dx} I^{(1-\mu)(1-\alpha)} f \right)(x). \tag{2.50}$$

Clearly $\mu = 1$ gives the Caputo derivative, while $\mu = 0$ gives the Riemann-Liouville derivative. The relationship of different orders $\mu < \lambda$ is given by the following theorem.

Theorem 2.6.3. *Let $f : [a, b] \to \mathbb{R}$ where $a < b < -\infty$ be absolutely continuous. If there exists $0 \leq \beta \leq 1 - \alpha$ then*

$$D^{\alpha,\mu} f(x) = D^{\alpha,\lambda} f(x) \tag{2.51}$$

whenever $0 \leq \mu < \lambda < (1 - \alpha - \beta)/(1 - \alpha)$. Further

$$D^{\alpha,\mu} f(x) = D^{\alpha,\lambda} f(x) + \frac{I^{\beta} f(a)}{\Gamma(1 - \alpha - \beta)(x - a)^{\alpha+\beta}} \tag{2.52}$$

holds whenever $\mu < \lambda = (1 - \alpha - \beta)/(1 - \alpha)$

Proof. The conditions are guaranteed from Samko [39] p39,

$$\frac{d}{dx} I^{1-\alpha} f(x) = \frac{f(a)}{\Gamma(1 - \alpha)(x - a)^{\alpha}} + I^{1-\alpha} \frac{d}{dx} f(x). \tag{2.53}$$

Letting $g(x) = I^{(1-\lambda)(1-\alpha)} f(x)$ and using the semi-group property

$$\begin{aligned} D^{\alpha,\lambda} f(x) &= I^{\lambda(1-\alpha)} \frac{d}{dx} I^{(1-\lambda)(1-\alpha)} f(x) \\ &= I^{\mu(1-\alpha)} I^{(\lambda-\mu)(1-\alpha)} \frac{d}{dx} g(x) \\ &= I^{\mu(1-\alpha)} \left(\frac{d}{dx} I^{(\lambda-\mu)(1-\alpha)} g(x) - \frac{(x - a)^{(\lambda-\mu)(1-\alpha)-1}}{\Gamma((\lambda - \mu)(1 - \alpha))} g(a) \right) \\ &= D^{\alpha,\mu} f(x) - \frac{(x - a)^{\lambda(1-\alpha)-1}}{\Gamma(\lambda(1 - \alpha)} g(a) \end{aligned} \tag{2.54}$$

where the fractional integral of the monomial was used. Then setting $g(a) = 0$ gives the first result, while the second result follows from $\lambda = (1 - \alpha - \beta)/(1 - \alpha)$. □

Before leaving this topic, let us note (without proof) the Laplace transform of the Hilfer fractional derivative:

$$\mathcal{L}\{D^{\alpha,\beta} f\} = s^{\alpha} \tilde{f}(s) - s^{\beta(\alpha-1)} D^{(1-\beta)(\alpha-1),0} f(0+) \tag{2.55}$$

when $0 < \alpha < 1$.

Chapter 3

Distribution Theory

In physics, many physical properties of substances (such as charge and mass) have no meaning on individual points. One can only approximate by taking ever shrinking neighbourhoods. Generalized functions, such as the Schwartz distributions below, are an attempt to construct mathematical objects which are more general than continuous functions as a solution to this dilemma. The theory is mainly due to Schwartz [41] (which is also where the term "distribution" originates), however, we will give an overview of Sobolev spaces, of which distributions have their origin. As this is a paper devoted to fractional calculus, we only consider the relavent properties here: the Fourier transform (which is of prime importance in applications), and the convolution (which will be used later to construct fractional integrals on the half line).

3.1 Sobolev space

Let $\Omega \subset \mathbb{R}$ be a bounded domain and let $f : \Omega \to \mathbb{R}$. Consider a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ for all $x \notin \Omega$, and φ has continuous derivatives up to an integer k. Then integration by parts

$$\int_{\Omega} \frac{d^k f}{dx^k}\, \varphi(x)\, dx = \int_{\Omega} (-1)^k f(x) \frac{d^k \varphi}{dx^k}\, dx. \tag{3.1}$$

is justified provided f also have continuous derivatives. Now suppose that we know nothing about the derivatives of f, but none the less, there exists a function

g such that

$$\int_{\Omega} f(x)\frac{d^k\varphi}{dx^k}\,dx = \int_{\Omega} (-1)^k \varphi(x)\, g(x)\, dx \tag{3.2}$$

for every such $\varphi(x)$. Then we say *g is a weak derivative of order k of f.* Note that "f has weak derivatives" does not coincide with "f has derivatives almost everywhere".

Definition 3.1.1. The space of all locally integral functions $f : \Omega \to \mathbb{R}$ which have weak derivatives of order k and integrable to order $p > 1$,

$$\frac{d^k f}{dx^k} \in L_p$$

is denoted $W_p^{(k)}$. The space $L_p^{(k)}$ will be the quotient of $W_p^{(k)}$ where $f = g$ if $d^k f/dx^k = d^k g/dx^k$.

Functions belonging to the same class of $L_p^{(k)}$ are called "mutually equivalent". $L_p^{(k)}$ is a normed space given by

$$\|f\|_{L_p^{(l)}} := \left(\int_{\Omega} \left(\frac{d^k f}{dx^k} \right)^p dx \right)^{1/p}. \tag{3.3}$$

It also turns out $W_p^{(k)}$ is also normed space whose norm is derived from a projection on polynomials. See Sobolev [42].

The power of the Sobolev spaces stem from the embedding theorems which give sufficient conditions as to when the weak solution to a differential equation is equal to the (strong) classical solution. Proofs can be found in Sobolev [42].

Theorem 3.1.2. *If $f \in W_p^{(k)}$ with $kp > 1$, then f is a continuous function. Further*

$$\|f\|_\infty := \max_{\Omega} |f(x)| \leq M \|f\|_{W_p^{(k)}} \tag{3.4}$$

for some constant M independent of f. Moreover, if $f \in W_p^{(k)}$ with $kp \leq 1$, then $f \in L_q$ where $q < p/(1 - kp)$.

Theorem 3.1.3. *Let $kp > 1$, then for any $K \subset W_p^{(k)}$ bounded by the norm of $W_p^{(k)}$, the image of K under the embedding to $C(\Omega)$ is compact. Moreover, if $kp \leq 1$, then the imbedding to L_q is strongly compact.*

3.2 Test functions on $\mathbb{R}$: the space $\mathcal{D}(\Omega)$

Whenever Paul Dirac used the delta function $\delta(x)$, he always did so with respect to some other function, $\int_{-\infty}^{\infty} \delta(x)\,\varphi(x)\,dx = \varphi(0)$. The function φ is some representative function, without which the symbol $\delta(x)$ would be meaningless. The delta function is an action of the function φ. This is analogous to our intuitive concept of a function. If we wish to make the symbol meaningful, we need to pick a representative point (say x) and define the function under its image (e.g. $f(x) = x^2$). In a similar vein, the function φ is the argument on which δ acts, and we define $\delta(\varphi) = \varphi(0)$. Thus the function space on which δ acts is the domain, and evaluated number $\varphi(0)$ is the range. Thus if we can properly define the function space, and restrict the generalized functions to it, then there are no problems. These spaces (and there are many we could choose) are called *fundamental spaces*, the individual functions are called *test functions*.

Motivated by Sobolev, we will introduce an important function space, the smooth (possessing continuous derivatives of all orders) functions with compact support.

Definition 3.2.1. Let $\Omega \subset \mathbb{R}$ be a domain. Then $E \subset \mathbb{R}$ is said to be *compact in* Ω if $\bar{E} \subset \Omega$ and $\bar{E}$ is a compact subset.

Definition 3.2.2. Let Ω be an open subset of $\mathbb{R}$. We define

$$\mathcal{D}(\Omega) := \{f \in C^\infty(\Omega) \mid \operatorname{supp} f \text{ is compact in } \Omega\}$$

where $\operatorname{supp} f$ is the set of $x \in \mathbb{R}$ such that $f(x) \neq 0$.

The size of the space is dependent on how large Ω is. Obviously, we can choose $\Omega = \mathbb{R}$, and thus obtain the largest possible space. This is the most important space in distribution theory. Observe that $\mathcal{D}(\Omega)$ is smaller than the test functions defined in Sobolev spaces as we require *all* derivatives to exist and be continuous.

However, it is not obvious that there even are non-trivial smooth functions with compact support. An example is

$$\omega_k(x) = \begin{cases} e^{-k^2/(k^2-x^2)} & \text{for } |x| < k \\ 0 & \text{for } |x| \geq k. \end{cases} \tag{3.5}$$

3.3 Distributions on $\mathbb{R}$: the space $\mathcal{D}'(\Omega)$

Distributions are in fact *functionals* on the space of test functions. That is, they take functions as arguments and have scalar values. If $\mathcal{K}$ is some fundamental space, then the set of distributions is defined as the *dual* $\mathcal{K}'$ (the set of bounded linear maps from $\mathcal{K} \to \mathbb{R}$). Then the most important distribution space is $\mathcal{D}'(\Omega)$. For example the delta distribution is defined as $\delta(\varphi) = (\delta, \varphi) := \varphi(0)$ for all $\varphi \in \mathcal{D}(\Omega)$. The notation (f, φ) is the convention used in the literature where f is distribution, φ is some test function, and (f, φ) is the scalar evaluation.

Observe that continuous functions are also elements of $\mathcal{D}'(\Omega)$. This because every $\varphi \in \mathcal{D}(\Omega)$ has compact support and so

$$\int_{-\infty}^{\infty} f(x)\,\varphi(x)\,dx < \infty \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{3.6}$$

In fact, as long as f is *locally integrable*, the expression is well defined. This is why the term "generalized function" is used: it really does generalize our concept of function. A distribution which is generated by a function using (3.6) is called *regular*, while one that is not is called *singular*.

Still, the real power of distributions lies in their concept of differentiation. Suppose that $f \in C^1(\mathbb{R})$, then by integration by parts,

$$\int_{-\infty}^{\infty} f'(x)\,\varphi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\,\varphi'(x)\,dx \tag{3.7}$$

because φ vanishes at infinity. This suggests how to differentiate these objects.

Definition 3.3.1. Let $\varphi \in \mathcal{D}(\Omega)$ and let $f \in \mathcal{D}'(\mathbb{R})$. Then the (generalized) *derivative* of f (written f') is defined to be $(f', \varphi) := -(f, \varphi')$.

From the discussion above, this is consistent with the case of differentiable functions. In general, a continuous function does not have a classical derivative, but it will always have a generalized derivative. In this sense, distributions generalize the concept of derivative.

Clearly, distributions form a vector space in its own right:

$$\begin{aligned}(f + g, \varphi) &= (f, \varphi) + (g, \varphi),\\ (\lambda f, \varphi) &= (f, \lambda\varphi),\end{aligned}$$

unfortunately we cannot form multiplication of distributions and still preserve associativity. This is because

$$(\delta \cdot x) \cdot \frac{1}{x} = 0 \cdot \frac{1}{x} = 0 \neq 1 = \delta \cdot 1 = \delta \cdot (x \cdot \frac{1}{x}).$$

We can however, form multiplication of a distribution with a smooth function: $(f\psi, \varphi) = (f, \psi \cdot \varphi)$ and the product of two smooth functions is again a smooth function.

The topology on $\mathcal{D}(\Omega)$ is defined through the norm

$$\|\varphi\|_{p,\Omega} \coloneqq \sup_{0 \le r} \sup_{x \in \Omega} |\varphi^{(r)}(x)|. \tag{3.8}$$

From this definition, a test function converges to zero if a function converges to zero uniformly together with all derivatives. For the dual space, we $f_n \in \mathcal{D}'(\Omega)$ converges to $f \in \mathcal{D}'(\Omega)$ if for any test function $\varphi \in \mathcal{D}(\Omega)$, $\lim_{n\to\infty}(f_n, \varphi) = (f, \varphi)$.

Example 3.3.2. Let $H(x)$ denote the Heaviside step function, which is zero for x negative and 1 when x is positive. The value of H at 0, is usually chosen to be 0, 1/2 or 1, but here we just define $H(0) = c_0$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} H'(x)\varphi(x)\,dx &= -\int_{-\infty}^{\infty} H(x)\varphi'(x)\,dx \\
&= -\int_{0}^{\infty} \varphi'(x)\,dx \\
&= -\varphi(x)\Big|_0^{\infty} = \varphi(0) = (\delta, \varphi).
\end{aligned}$$

Thus the Heaviside step function, which doesn't have a derivative at $x = 0$ does have δ as its generalized derivative.

Example 3.3.3. The distribution x_+^λ This is a very special distribution and we will do a systematic study of it here. Let us follow Gel'fand and Shilov [13] in observing that the function

$$x_+^\lambda = \begin{cases} x_+^\lambda & \text{for } x \le 0 \\ 0 & \text{for } x > 0 \end{cases} \tag{3.9}$$

defines a regular distribution

$$(x_+^\lambda, \varphi) = \int_0^\infty x^\lambda \, \varphi(x)\, dx, \tag{3.10}$$

for real $\lambda > -1$. In fact, this is true for complex λ if $\operatorname{Re}\lambda > -1$. Our first aim is use analytic continuation to define (x_+^λ, φ) (which is a function of λ) for other values of λ. We can rewrite (3.10) as

$$(x_+^\lambda, \varphi) = \int_0^1 x^\lambda(\varphi(x) - \varphi(0))\, dx + \int_1^\infty x^\lambda \varphi(x)\, dx + \frac{\varphi(0)}{\lambda+1}. \tag{3.11}$$

The first term is defined if $\operatorname{Re}\lambda > -2$ and the third term is defined if $\operatorname{Re}\lambda \neq -1$. Thus we have arrived at a continuation of (x_+^λ, φ) if $\operatorname{Re}\lambda > -2, \lambda \neq -1$. To obtain expressions $\operatorname{Re}\lambda > -n, \lambda \neq -1, -2, \ldots, -n$ we define

$$\int_0^\infty x^\lambda \varphi(x) dx = \int_0^1 \left[\varphi(x) - \varphi(0) - x\varphi'(0) - \ldots - \frac{x^{n-a}}{(n-1)!}\varphi^{(n-1)}(0)\right] dx \tag{3.12}$$

$$+ \int_1^\infty x^\lambda \varphi(x)\, dx + \sum_{k=1}^{n} \frac{\varphi^{(k-1)}(0)}{(k-1)!(\lambda+k)}. \tag{3.13}$$

This function $(x_+^\lambda, \varphi)(\lambda)$ is analytic on the complex plane except where it has poles when $\lambda = -1, -2, \ldots$. To remove these poles, we normalize the function by dividing by a similar function with poles at the same points. The residue of x_+^λ is $(-1)^{n-1}\delta^{(n-1)}(x)/(n-1)!$. Consider

$$(x_+^\lambda, e^{-x}) = \int_0^\infty x^\lambda e^{-x} dx = \Gamma(\lambda+1), \tag{3.14}$$

then we can divide by $\Gamma(\lambda+1)$ to obtain

$$\begin{aligned} \left.\frac{x_+^\lambda}{\Gamma(\lambda+1)}\right|_{\lambda=-n} &= \frac{\operatorname*{res}_{\lambda=-n} x_+^\lambda}{\operatorname*{res}_{\lambda=-n} (x_+^\lambda, e^{-x})} \\ &= \frac{(-1)^{n-1}\delta^{(n-1)}(x)/(n-1)!}{(-1)^{n-1}(\delta^{(n-1)}(x), e^{-x})/(n-1)!} = \delta^{(n-1)}(x) \end{aligned} \tag{3.15}$$

because $(\delta^{(n-1)}(x), e^{-x}) = 1$. Thus $(x_+^{-n}/\Gamma(1-n), \varphi) = \varphi^{(n-1)}(x)$.

3.4 Tempered distributions and the Fourier transform

Possibly the most important operation in operational calculus (solving differential equations) is the Fourier transform. Suppose that f is regular, then by Fubini's Theorem

$$\begin{aligned}
(\hat{f}, \varphi) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx \right) \varphi(\omega)\, d\omega \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega x} f(x) \varphi(\omega)\, dx\, d\omega \\
&= \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} e^{i\omega x} \varphi(\omega)\, d\omega \right) dx \\
&= (f, \hat{\varphi}).
\end{aligned} \tag{3.16}$$

Thus we can formally define $(\hat{f}, \varphi) := (f, \hat{\varphi})$ for arbitrary distributions. However, in general, $\hat{\varphi}$ will not have compact support. There are two options to rectify this problem:

- We enlarge the space of test functions to allow for this situation.
- We look at the image of $\mathcal{F}$ and define a whole new space.

The second option leads to the concept of *ultradistributions*, denoted $\mathcal{Z}'(\mathbb{R})$. It is a nice space, with the Fourier transform acting as an isomorphism from $\mathcal{D}'(\mathbb{R})$ to $\mathcal{Z}'(\mathbb{R})$. However, this takes us outside the scope of this paper, and we thus focus on Case 1. The interested reader can see Gel'fand and Shilov [13].

Definition 3.4.1. The space of *Schwartz test functions of rapid decrease* is the set of infinitely differentiable functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that for all natural numbers n and r,

$$\lim_{x \to \pm\infty} |x^n \varphi^{(r)}(x)| = 0.$$

In words, φ converges to zero faster than any polynomial. This space is denoted by $\mathcal{S}(\mathbb{R})$. Most authors drop the $\mathbb{R}$ as Schwartz's space is only ever used over the whole real line. From an applications point of view, especially from the perspective of physicists, $\mathcal{S}'$ is the natural space to study, as Fourier analysis is in some sense "closed" in this space.

We introduce in $\mathcal{S}$ a countable number of norms via

$$||\varphi||_p = \sup_{n\leq p}(1+|x|^2)^{p/2}|\varphi^{(n)}|, \quad \varphi \in \mathcal{S}, p = 0, 1, 2, \ldots \tag{3.17}$$

and define convergence on $\mathcal{S}'(\mathbb{R})$ by (φ_n) converges to φ if it converges on every norm. It is clear that $\mathcal{D}'(\mathbb{R}) \subset \mathbb{R}$ however the function e^{-x^2} is an element of the latter but not the former. Yet $\mathcal{D}'(\mathbb{R})$ is dense in $\mathcal{S}'(\mathbb{R})$, that is, for any $\varphi \in \mathcal{S}'(\mathbb{R})$ there is a sequence $(\varphi_n) \subset \mathcal{D}'(\mathbb{R})$ which converges to φ in $\mathcal{S}'(\mathbb{R})$. Indeed, define

$$\eta(x) = 1 \text{ for } |x| < 1$$

and then the functions

$$\varphi_n(x) = \varphi(x)\,\eta(\frac{x}{n}), \quad n = 1, 2, \ldots$$

converge to φ. In fact $\mathcal{S}(\mathbb{R})$ is dense in $\mathcal{S}'(\mathbb{R})$, see Vladimirov [43].

The most remarkable property of the class of tempered distributions is that the Fourier transform of a tempered function is also a tempered function. Since the tempered functions $\varphi \in \mathcal{S}(\mathbb{R})$ are integrable on $\mathbb{R}$, the classical operation of the Fourier transform

$$\mathcal{F}\varphi(x) = \int_{-\infty}^{\infty} \varphi(t)\, e^{itx}\, dt \tag{3.18}$$

is well defined, and $\mathcal{F}\varphi$ is bounded (because φ is bounded) and continuous in $\mathbb{R}$. Further, because φ decreases to zero faster than any ploynomial, we can differentiate under the integral sign to obtain

$$\begin{aligned} \frac{d^n}{dx^n}\mathcal{F}\varphi(x) &= \int_{-\infty}^{\infty} (it)^n\, \varphi(t)\, e^{ixt}\, dt \\ &= \mathcal{F}\{(it)^n\varphi(t)\}(x) \end{aligned} \tag{3.19}$$

and so $\mathcal{F}\varphi$ is a smooth function.

Theorem 3.4.2. *The Fourier transform is a linear continuous injection of* $\mathbb{R}$ *to itself.*

Proof. Let $\varphi \in \mathcal{S}'(\mathbb{R})$ We have

$$\begin{aligned}(1+|x|^2)^{p/2}\left|\frac{d^n}{dx^n}\mathcal{F}\varphi(t)\right| &\leq (1+|x|^2)^{[\frac{p+1}{2}]}|\frac{d^n}{dx^n}\mathcal{F}\varphi(t)| \\ &\leq \left|\int_{-\infty}^{\infty}(1-K)^{[\frac{p+1}{2}]}[(it)^n\varphi(t)]e^{ixt}dt\right| \\ &\leq C\sup(1+|x|^2)^{(n+1)/2}\left|(1-K)^{[\frac{p+1}{2}]}|x^n\varphi(x)|\right|,\end{aligned}$$

and thus we obtain estimates

$$\|\mathcal{F}\varphi\|_p \leq C\|\varphi\|_{p+n+1}, \quad p=0,1,\ldots \tag{3.20}$$

for a constant C that does not depend on φ. Bounded operators are continuous, so FTO is continuous. Further because $\mathcal{F}^{-1}\psi(-x) = 1/2\pi\,\hat{\psi}(-x)$, the inverse Fourier transform is continuous. Lastly if $\psi = \mathcal{F}^{-1}\varphi$ then $\mathcal{F}^{-1}0 = 0$ so the Fourier transform is injective. □

3.5 The convolution of distributions

If $f(x)$ and $g(x)$ are two square integrable functions, then one can define their convolution as

$$f * g(x) := \int_{-\infty}^{\infty} f(u)g(x-u)du = \int_{-\infty}^{\infty} f(x-u)g(u)du. \tag{3.21}$$

This operation is always defined in f and g both have compact support, and has important applications in optics where it represents a "moving average" of f with g. In general though, the convolution may fail to exist. The analogous operation for distributions, is probably of even more importance, and we shall use it in section 4.1. The properties of the convolution is inherently tied with the *direct product* (also known as the *tensor product*) thus we will make a systematic study of it first.

Definition 3.5.1. Let $f(x)$ be a distribution in $\mathcal{D}'(\Omega)$ and $g(y)$ be a distribution in $\mathcal{D}'(\Omega)$, the two variable distribution $f \oplus g \in \mathcal{D}'(\Omega \times \Omega)$ is defined as:

$$(f \oplus g, \varphi(x,y)) := (f(x),(g(y),\varphi(x,y))) \tag{3.22}$$

Now if $\varphi \in \mathcal{D}'(\Omega)$ and $\psi \in \mathcal{D}'(\Omega)$, then their product function $\varphi(x)\psi(y)$ is an element of $\mathcal{D}'(\Omega)$. Then $(f(x) \oplus g(y), \varphi(x)\psi(y)) = (f(x), \varphi(x)(g(y), \psi(y))) = (f, \varphi) \cdot (g, \psi)$. Similarly $(g(y) \oplus f(x), \varphi(x)\psi(y)) = (g, \psi) \cdot (f, \varphi)$ so the direct product is commutative. It is also associative. We then use this definition to define the convolution.

Definition 3.5.2. Let f and g be as above, then the *convolution* $f * g$ is

$$(f * g, \varphi) = (f(x) \oplus g(y), \varphi(x + y)) = (f(x), (g(y), \varphi(x + y))) \tag{3.23}$$

The convolution is commutative because the direct product is. The problem with this simple definition is that in general $\varphi(x+y)$ does not have compact support in $\Omega \times \Omega$ and as such, not in $\mathcal{D}(\Omega \times \Omega)$. However, we can give the definition meaning if we consider distributions bounded support: we say that a distribution *vanishes* in an open set $E \subset \Omega$ if $(f, \varphi) = 0$ for all $\varphi \in \mathcal{D}(E)$. The complement $\Omega \setminus E$ is called the *support of f*.

Then the convolution definitely exists for the following three cases:

- One of f or g has compact support.
- Both are bounded on the left: support is a half interval (a, ∞)
- Both are bounded on the right: support is a half interval $(-\infty, b)$

Note that if Case 1 holds, then both are bounded from below by $c = \inf \operatorname{supp} f \cup \operatorname{supp} g$. So, Case 1 is subsumed in Case 2. For Case 3, it is Case 2 but with the ordering switched. We will prove only Case 2.

Proof. Assume that the supports of f and g are bounded on the left. Then consider $(f(x), \varphi(x + y))$. This is clearly infinitely differentiable in y. Let y be very large, then the support of $(f(x), \varphi(x + y))$ does not intersect that of $f(x)$. Thus for such y, $(f(x), \varphi(x + y)) = 0$. This implies the support of $(f(x), \varphi(x + y))$ is bounded on the right, and also on the left (by hypothesis). Thus the definition of convolution well-defined. □

If one of the above holds then the convolution exists and is associative.

Chapter 4

Fractional Calculus of Distributions

4.1 Convolution Method

The method via a convolution is due to Schwartz [41] and works very nicely on the half-axis. Recall the definition of the generalized function x_+^λ with its normalization (Example 3.3.3). Then the Riemann-Liouville fractional integration can be rewritten as a convolution

$$\frac{1}{\Gamma(\lambda)} \int_0^x f(u)\,(x-u)^{\lambda-1}\,du = f(x) * \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}. \tag{4.1}$$

We define the distribution 4.1 as the *primitive of order* λ of f. The definition holds simply if $\operatorname{Re}\lambda > 0$ while for other values it is the regularization. This definition holds if convolution is valid. If we restrict f to the real line, then both distributions are bounded from the origin and thus the convolution is consistent. We will follow Gel'fand & Shilov [13] in denoting

$$\Phi_\lambda = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}.$$

Then because $\Phi_{-n} = \delta^{(n)}(x)$ for positive integers by equation (3.15), we can write

$$\begin{aligned} f(x) * \Phi_0 &= f(x) * \delta(x) = f(x), \\ f(x) * \Phi_{-1} &= f(x) * \delta'(x) = f'(x), \end{aligned}$$

and so on. So we obtain fractional derivatives as well. Further, the operator satisfies the semigroup property $\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu}$. Indeed if $\operatorname{Re}\lambda, \operatorname{Re}\mu > 0$ then

$$\Gamma(\lambda)\,\Gamma(\mu)\,\Phi_\lambda * \Phi_\mu = \int_0^x u^{\lambda-1}\,(x-u)^{\mu-1}\,du.$$

Substituting $u = xt$ to the right hand side gives

$$x^{\lambda+\mu-1}\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}dt = x^{\lambda+\mu-1}B(\lambda,\mu) = x^{\lambda+\mu-1}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}.$$

Which proves our result. Other values of λ and μ can can be proved using analytic continuation.

It is erroneously stated in [39] that for $\lambda = -n$, we must replace with standard differentiation. The definition is consistent for all complex λ.

Example 4.1.1. The most cited example of an application of fractional calculus is Abel's integral equation, which is related to the tautochrone problem. We will derive this equation in Section 5.1 but here we will use distribution theory to solve it. For a elementary derivation of the solution see Gorenflo and Vessella [15]. The equation is defined as

$$g(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(u)}{(x-u)^{1-\alpha}}du, \tag{4.2}$$

where $g(x)$ is a given function, and we wish for a unique solution. In the classical theory one assumes $0 < \alpha < 1$ as this guarantees the convergence of the integral. A derivation of the solution in the space of integrable functions is given in [39]. Here we do not place constraints in α as we use the convolution

$$g(x) = f(x) * \Phi_\alpha,$$

then to find an expression for f

$$f(x) = f(x) * \delta(x) = f(x) * \Phi_\alpha * \Phi_{-\alpha} = g(x) * \Phi_{-\alpha}. \tag{4.3}$$

If we assume that $0 < \alpha < 1$ and $g(x)$ to be differentiable, then

$$f(x) = g(x) * \Phi_{-\alpha} = g'(x) * \Phi_{1-\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{g'(u)}{(x-u)^\alpha}\,du, \tag{4.4}$$

which agrees with the elementary solution. Note that in [39] the answer given has a $g(0)$ term, but if g has positive support and still be continuous then $g(0) = 0$.

4.2 Multiplier Method (Tempered Distributions)

In this section we consider fractional integration as defined through Fourier transforms. It seems natural to take $\mathcal{S}(\mathbb{R})$ the space of Schwartz test function (Section 3.4) because the Fourier transform is invariant on it, but in fact the requirement that we have rapid vanish at infinity is too strong. For example

$$(I_{+}^{\alpha}\varphi)(x) \geq \frac{1}{\Gamma(\alpha)} \int_{a}^{b} (x-u)^{\alpha-1}\,\varphi(u)\,du \tag{4.5}$$

$$\geq \min_{a\leq u\leq b} \varphi(u) \frac{(x-b)^{\alpha}-(x-a)^{\alpha}}{\Gamma(\alpha+1)} \tag{4.6}$$

and when $x \to \infty$, $(I_{+}^{\alpha}\varphi)(x) \sim Kx^{\alpha-1}$. This diverges if $\alpha \geq 1$. This is occuring because the Reimann-Liouville differintegral treats negative infinity and positive infinty differently, the negative infinity is in the limits of the integral, but we must treat $x \to \infty$ seperately. One option is to consider *two* spaces each which is invariant to I_{+}^{α} and I_{-}^{α} respectively. However, this means that power functions do not exist in the duals of these spaces.

We follow Lizorkin [23] in constructing a space $\Phi \subset S$ which is invariant withh respect to fractional calculus. Now the fractional integral with respect to Fourier transform is

$$\mathcal{F}(I_{\pm}^{\alpha}\varphi)(\omega) = (\mp i\omega)^{-\alpha}\hat{\varphi}(\omega) \tag{4.7}$$

where I_{-} refers to the bottom limit $a = -\infty$ while I_{+} refers to the upper limit $b = \infty$. Now if the function $\hat{\varphi}(\omega)$ is not worse after multiplication by $(\mp i\omega)^{-\alpha}$ then our space is invariant. Thus we define the function space

$$\Psi = \{\varphi \in \mathcal{S}(\mathbb{R}) \,|\, \varphi^{(n)}(0) = 0,\ n \in \mathbb{N}_0\} \tag{4.8}$$

to be the functions which rapidly vanish at $\pm\infty$ and at zero together with all derivatives. These functions are smooth but not analytic. An example is $\varphi(x) = e^{-x^2-x^{-2}}$. We define *Lizorkin's space* to be the preimage of the Fourier transform of Ψ:

$$\Phi = \{\varphi \in \mathcal{S}(\mathbb{R}) \,|\, \hat{\varphi} \in \Psi\}. \tag{4.9}$$

Then,

$$0 = \hat{\varphi}^{(k)}(0) = \int_{-\infty}^{\infty} e^{i0t}\varphi(x)x^{n}dx, \tag{4.10}$$

implies that $\Phi(\mathbb{R})$ is the space of function which are orthogonal to the polynomials $\mathcal{P}$.

Theorem 4.2.1. *Equation 4.7 is valid if $\varphi \in \Phi(\mathbb{R})$ and if $\alpha \geq 0$.*

Proof. *Case:* $0 < \alpha < 1$. This given in the classical Fourier transform from above.

Case: $1 < \alpha < 2$.

$$
\begin{aligned}
(\mathcal{F} I_+^\alpha \varphi)(\omega) &= \frac{1}{\Gamma(\alpha)} \lim_{b\to\infty} \int_{-\infty}^{b} \left(\int_{-\infty}^{x} e^{i\omega x}(x-u)^{\alpha-1}\varphi(u)\, du\, dx \right) \\
&= \frac{1}{\Gamma(\alpha)} \lim_{b\to\infty} \left(\int_{-\infty}^{b} \int_{0}^{b-u} \varphi(u) e^{i\omega u} \xi^{\alpha-1} e^{i\omega\xi}\, d\xi\, du \right) \\
&= \frac{1}{i\omega\Gamma(\alpha)} \lim_{b\to\infty} \left(b^{\alpha-1} e^{i\omega b} \int_{-\infty}^{b} (1-u/b)^{\alpha-1} \varphi(u) du \right. \\
&\qquad \left. - (\alpha-1) \int_{-\infty}^{b} \int_{0}^{b-u} \xi^{\alpha-2} e^{i\omega\xi}\, d\xi\, du \right).
\end{aligned} \tag{4.11}
$$

The first vanishes by L'Hôpital's rule and equation (4.10) while the second can be evaluted by limit passage. Thus

$$
(\mathcal{F} I_+^\alpha \varphi)(\omega) = \frac{(\alpha-1)\Gamma(\alpha-1)}{(-i\omega)^\alpha \Gamma(\alpha)} \hat{\varphi}(\omega). \tag{4.12}
$$

Observe that $\Gamma(x) = (x-1)\Gamma(x-1)$ and so (4.12) is equal to $\hat{\varphi}(\omega)/(-i\omega)$.

Case: $\alpha = 1$

$$
(\mathcal{F} I_+^1 \varphi)(\omega) = \lim_{b\to\infty} \int_{-\infty}^{b} \varphi(u) \frac{e^{i\omega b} - e^{iu\omega}}{i\omega} du = -\frac{1}{i\omega} \hat{\varphi}(\omega).
$$

by Equation 4.10.

Case: $\alpha > 1$. This is nothing more than combining the above cases with the semigroup property of fractional integrals. □

In actual fact, the theorem holds for complex α, $\operatorname{Re} \alpha > 0$, however, the proof requires theory of L^p spaces and thus we will not consider it.

From the above theorem, the Lizorkin space $\Phi(\mathbb{R})$ is a space of test functions which is dense in $\mathcal{S}(\mathbb{R})$ In fact, $\Phi(\mathbb{R})$ is $\mathcal{S}'(\mathbb{R})/\mathcal{P}$, that is two Lizokin distributions are equal if they differ by a polynomial. We define fractional calculus on $\Phi(\mathbb{R})$ by

$$
(I_\pm^\alpha f, \varphi) := (f, I_\mp^\alpha \varphi), \;\; \varphi \in \Phi(\mathbb{R}). \tag{4.13}
$$

Interestingly, despite its elegance, Lizokin's Space doesn't seem to be well known. For example, Hilfer [18] uses fractional calculus on Schwartz distributions but does not consider $\mathbb{R}$. Nonetheless it is used in Ribin et al. [36] to study the wavelet representations of fractional integration.

Chapter 5

Selected Applications

5.1 Recent formulation of Abel's mechanical problem

The first physical application of fractional integration was given by Abel [1] in his solution to the tautochrome problem. In Example 4.1.1 we solved the equation, but here we will derive the equation by using a reformulation due to Keller [19].

Consider a friction-less hill which is monotonically increasing. Without loss of generality, we will assume the hill is cenetred at the origin. Our goal is to model a particle with mass $m > 0$ along the hill at time $t = 0$, and trace out the the path of the paticle. We will denote the hill be $y(x)$ with $y(0) = 0$ and assume $y(x)$ is differentiable.

Let v_0 be the initial velocity of the particle at $t = 0$, then the kinetic energy at this time is $E = mv_0^2/2$. Let g be the acceleration due to gravity. Then its potential energy at (x, y) is $V = mgy$. We paramtrize the hill by using arclength s with $x(0) = y(0) = 0$, and so $V(s)$ is also strictly increasing and $V(0) = 0$. So the inverse function $s(V)$ exists.

With $\boldsymbol{s}(t)$ being the position at time t, and thus it satisfies the DE

$$m\frac{d^2\boldsymbol{s}}{dt^2} + \frac{dV(s)}{d\boldsymbol{s}} = 0. \tag{5.1}$$

Multiplying by $\boldsymbol{s}'(t)$ and integrating we obtain

$$\frac{m}{2}\boldsymbol{s}'(t)^2 + V(\boldsymbol{s}(t)) = E = \frac{mv_0^2}{2}, \tag{5.2}$$

whose solution is

$$t = \sqrt{m/2} \int_0^s \frac{du}{\sqrt{E - V(u)}}, \tag{5.3}$$

as long as $V(s) \leq E$. Because $s(V)$ is the inverse to $V(s)$, if it is finite it exists, so

$$T(E)/2 = \sqrt{m/2} \int_0^{s(E)} \frac{du}{\sqrt{E - V(u)}}. \tag{5.4}$$

This is finite only if $V'(s(E)) > 0$. Otherwise, the particle never returns. Let us choose E_0 such that for $V \leq E_0$, the inverse function $s(V)$ exists. Then we obtain

$$T(E) = \sqrt{2m} \int_0^{s(E)} \frac{du}{\sqrt{E - V(u)}}. \tag{5.5}$$

By changing variables

$$\frac{T(E)}{\sqrt{2m}} = \int_0^E \frac{s'(V)\,dV}{\sqrt{E - V}} \quad \text{for } 0 \leq E \leq E_0. \tag{5.6}$$

This is the standard Abel integral equation which first appeared. There are many other examples of fractional calculus used in mechanics, see Mainardi [25].

5.2 Stereology of spheres

In applied sciences, one is commonly confronted with the problem is determining the size distribution of particles in an opaque medium. Normally one only counts (using statical methods or otherwise) a single plane cut, however, the observed radii is different to the true radii because the plane misses the centers of most of the spheres. This problem occurs in, among others, corpuscles embedded in tissue [45] and crystallography [33].

Let us construct the problem as follows. Consider spherical particals randomly distributed in $\mathbb{R}^3$ i.e. the centers are distributed in space according the the Poisson distribution and the spacial density is unknown. Further, let the radii follow a

probability distribution $f(r)$ which is *unknown*, except that it is bounded by some $R < \infty$. Let λ be the mean number of centers of radius 1. Then we should have

$$f(r) \geq 0 \text{ and } \int_0^R f(r) = 1, \tag{5.7}$$

provided that we assume the spheres are spread out so that the mean distance between centers is much larger than the mean radius $0 < r_0 < \infty$ (i.e. there are no collisions).

Suppose we intersect $\mathbb{R}^3$ with a plane E. This will intersect some of the spheres, and each sphere will show up as a circle. The apparent radius x however will not equal the actual radius r because only rarely will E pass through the center of the spheres. Nonethless, the x is distributed according to some probability distribution $g(x)$ with $0 \leq x \leq R < \infty$ and whose integral equals 1. Now we estimate g as close as we like (by using the conventional statistical methods), but we are faced with trying to solve $f(r)$.

The expected number of spheres with (actual) radius between r and $r + dr$ which intercepts E such that the center of the circle (in the cut) lies with an area of 1 is $\lambda \cdot 2r \cdot f(r)\,dr$. Thus the probability density $f^*(r)$ for a sphere intersecting E is $f^*(r) = rf(r)/r_0$.

Take a sphere with radius r and let $y \in [0, r]$. The probability density of its center having a distance y from E is piecewise constant: r^{-1} if $y \in [0, r]$ and zero elsewhere. If x is the apparent radius $y = \sqrt{r^2 - x^2}$ for $0 \leq x \leq r$, so given r, y is decreasing in x and $dy/dx = -x/\sqrt{r^2 - x^2}$. So the event that a sphere of actual radius r cutting E has apparent radius x, obeys the probabilty density

$$-\frac{1}{r}\frac{dy}{dx} = \frac{1}{r}\frac{x}{\sqrt{r^2 - x^2}} \quad 0 \leq x \leq r. \tag{5.8}$$

We multiply both sides by $f^*(r)$ to obtain

$$-\frac{f(r)}{r_0}\frac{dy}{dx} = \frac{x}{r_0}\frac{f(r)}{\sqrt{r^2 - x^2}}, \tag{5.9}$$

and then integrate over $x \leq r \leq R$ to obtain

$$g(x) = \frac{x}{r_0}\int_x^R \frac{f(r)}{\sqrt{r^2 - x^2}}dr \quad 0 \leq x \leq R. \tag{5.10}$$

Firstly, it is significant that this is a right-integral. The majority of applied fractional calculus (especially in mechanics) involve left integrals. There, the interpretation of the integral is an operation on events which have transpired. Right integrals however use future states of the particles, which are not observable most in physical problems. Secondly, the expression (5.10) is very similar to the classical fractional integral, except for the squares. Operators of this form are called Eridyle-Kober operators, and, as we shall see, are related the classical fractional integral. But first, we need to verify that this really is probility distribution:

$$\begin{aligned}\int_0^R g(x)dx &= \int_0^R \frac{x}{r_0} \int_x^R \frac{f(r)}{\sqrt{r^2 - x^2}}\, dr\, dx \\ &= r_0^{-1} \int_0^R f(r) \int_0^r \frac{x}{\sqrt{r^2 - x^2}}\, dx\, dr \\ &= r_0^{-1} \int_0^R r f(r)\, dr = 1. \end{aligned} \tag{5.11}$$

Well and good. Unfortunately our problem was given g, find f, so we need to invert (5.10). We first simplify by letting $\tilde{g}(x) = g(x)/x$ and $\tilde{f}(r) = f(r)/r_0$ then the (5.10) becomes

$$\frac{1}{\Gamma(1/2)} \int_x^R \frac{\tilde{f}(r)}{\sqrt{r^2 - x^2}} dr = \frac{\tilde{g}(x)}{\Gamma(1/2)}. \tag{5.12}$$

Let $\tau = r^2, \xi = x^2, \hat{g}(\xi) = \tilde{g}(\sqrt{x})/\Gamma(1/2)$ then $d\tau/dr = 2r$ and so $dr = d\tau/2\sqrt{\tau}$ and finally we let $v(\tau) = \tilde{f}(\sqrt{\tau})/2\sqrt{t}$. We now obtain

$$\frac{1}{\Gamma(1/2)} \int_x^{R^2} \frac{v(\tau)}{\sqrt{\tau - \xi}} d\tau = \hat{g}(\xi).$$

Now let $u = R^2 - \tau$, $du/d\tau = -1$,

$$\frac{-1}{\Gamma(1/2)} \int_0^{R^2-\xi} \frac{v(R^2 - u)}{\sqrt{R^2 - u - \xi}} du = \hat{g}(\xi)$$

Let $X = R^2 - \xi$ and $V(u) = v(R^2 - u)$ to get the expression

$$\frac{-1}{\Gamma(1/2)} \int_0^X \frac{V(u)}{\sqrt{X-u}} du = \hat{g}(R^2 - u). \tag{5.13}$$

Observe that this is a Riemann-Liouville integral of order 1/2. Inverting (5.13), we obtain

$$v(\tau) = \frac{-1}{\Gamma(1/2)} \frac{d}{d\tau} \int_\tau^{R^2} \frac{\hat{g}(\xi)}{\sqrt{\xi - \tau}} d\xi. \tag{5.14}$$

And resubstituting and using $\Gamma(1/2) = \sqrt{\pi}$

$$\frac{f(r)}{2rr_0} = \frac{-1}{\Gamma(1/2)} \frac{d}{dr^2} \int_r^R \frac{g(x)}{x\Gamma(1/2)} \frac{1}{\sqrt{x^2 - r^2}} \frac{d\xi}{dx} dx, \tag{5.15}$$

$$\frac{f(r)}{r_0} = \frac{-2}{\pi} \frac{d}{dr} \int_r^R \frac{g(x)}{\sqrt{x^2 - r^2}} dx. \tag{5.16}$$

This problem has been generalized by considering a horizontal strip instead of a full plane. Named the “Tomato Salad Problem”, this too reduces to an integral equation related to fractional calculus. The interested reader is invited to read Gorenflo [14].

5.3 Fractional Cauchy problem

As an example of fractional differential equations, the fractional Cauchy problem is defined as

$$D^{\alpha,\beta} f(t) = -Kf(t); \;\; I^{(1-\beta)(1-\alpha)} f(0+) = f_0 \tag{5.17}$$

where $D^{\alpha,\beta}$ is the Hilfer fractional derivative 2.6.2 with parameters $0 < \alpha < 1$, $0 \leq \beta \leq 1$, base point $a = 0$, $t \geq 0$ and K is a real constant. Applying the Laplace transform to both sides, one obtains

$$s^\alpha \tilde{f}(s) - s^{\beta(\alpha-1)} f_0 = -K\tilde{f}(s) \tag{5.18}$$

giving

$$\tilde{f}(s) = \frac{s^{\beta(\alpha-1)} f_0}{s^\alpha + K}. \tag{5.19}$$

To invert this, we define $\gamma = \alpha + \beta(1-\alpha)$ so that

$$\tilde{f}(s) = f_0 \frac{s^{\alpha-\gamma}}{s^\alpha + K} = s^{-\gamma} f_0 \frac{1}{1 - s^{-\alpha}(-K)} = f_0 \sum_{n=0}^{\infty} (-K)^n s^{-\alpha n - \gamma}, \tag{5.20}$$

$$f(t) = f_0\, x^{\gamma-1} \sum_{n=0}^{\infty} \frac{(-Kt^\alpha)^n}{\Gamma(\alpha n + \gamma)} = f_0\, t^{\gamma-1} E_{\alpha,\gamma}(-Kt^\alpha), \tag{5.21}$$

where the 2-parameter Mittag-Leffler function is defined as

$$E_{\mu,\lambda}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\mu\, n + \lambda)}. \tag{5.22}$$

For the properties of this important function see Podlubny [31].

5.4 Special functions as fractional derivatives

It should be well known that most special functions are either instances of the hypergeometric function, the Meijer G-function or the Fox H-function. Here we will establish some properties of the hypergeometric function, and later show that the Bessel function is the fractional derivative of an elemetrary function. We will use the distributions from the convolution method, although in theory we do not need to (see for example see Oldham and Spanier [29]).

First recall the definition of the hypergeometric function:

$$_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \tag{5.23}$$

where we have used the Pochhammer symbol $(a)_0 := 1$ and $(a)_n := \Gamma(a+n)/\Gamma(a)$. Many functions are special cases of the hypergeometric such as

$$\begin{aligned} e^x &= {}_2F_1(-,-;-;x), \\ \arctan x &= x {}_2F_1(-,-;3/2;-x^2/4), \\ J_\nu(x) &= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu {}_2F_0(-,-;\nu+1;-x^2/4). \end{aligned}$$

Of import here is the integral representation

$$_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-ux)^{-\alpha} du, \tag{5.24}$$

which is defined provided $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$ and $|x| < 1$. Let $\omega = ux$ in (5.24) so that

$$ {}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \frac{1}{x^{\gamma-1}} \int_0^x \omega^{\beta-1}(1-\omega)^{-\alpha}(x-\omega)^{\gamma-\beta-1} d\omega, \tag{5.25} $$

multiplying through by $\frac{x^{\gamma-1}}{\Gamma(\gamma)}$ yields

$$ \frac{x^{\gamma-1}}{\Gamma(\gamma)} {}_2F_1(\alpha,\beta;\gamma;x) = \int_0^x \frac{(x-\omega)^{\gamma-\beta-1}}{\Gamma(\gamma-\beta)} \frac{\omega^{\beta-1}(1-\omega)^{-\alpha}}{\Gamma(\beta)} d\omega \tag{5.26} $$

$$ = \Phi_{\beta-\gamma} * \frac{x_+^{\beta-1}(1-x)_+^{-\alpha}}{\Gamma(\beta)}. \tag{5.27} $$

a result found in Gel'fand and Shilov [13]. The power of using distributions lies in not having to justify for which values of the parameters the integral converges. To further illustrate the point, consider the integral representation of the Bessel function (itself a hypergeometric)

$$ J_\nu(x) = \frac{2(\frac{1}{2}x)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \int_0^1 (1-u^2)^{\nu-1/2} \cos xu \; du, \tag{5.28} $$

where ν is defined for $\operatorname{Re}\nu > -1/2$. For other values, we understand the function as the regularization of the inetgral. Setting $\omega = xu$ as before

$$ J_\nu(x) = \frac{2}{(2x)^\nu \Gamma(\nu+1/2)\sqrt{\pi}} \int_0^x (x^2-\omega^2)^{\nu-1/2} \cos\omega \; d\omega, \tag{5.29} $$

$$ 2^\nu \sqrt{\pi} u^{\nu/2} J_\nu(\sqrt{u}) = \int_0^u \frac{(u-y)^{\nu-1/2}}{\Gamma(\nu+1/2)} \frac{\cos\sqrt{y}}{\sqrt{u}} dy, \tag{5.30} $$

$$ = \Phi_{\nu-1/2} * \frac{\cos\sqrt{u}}{\sqrt{u}}. \tag{5.31} $$

which is a very pleasing result. Interestingly, Oldham and Spanier [29] argue it is more enlightening to introduce various special functions as fractional derivatives of elementary functions, rather than as integrals (such as the incomplete gamma function) or as solutions to differential equations (such as Airy's function).

Appendix A

Generalizations of Fractional Calculus

A.1 Erdélyi-Kober operators and McBride's Space

In Section 5.2, it was shown that integral equations where the integral kernel is expressed in terms of squares of the variables; that is, $(x^2 - t^2)^{\alpha-1}$ is related to the classical Abel integral equation (and hence to Riemann-Liouville fractional integrals). Motivated by these results, the Riemann-Liouville integral has been generalized as

$$I_m^\alpha f(x) = \frac{m}{\Gamma(\alpha)} \int_0^x (x^m - u^m)^{\alpha-1} u^{m-1} f(u)\, du \tag{A.1}$$

$$K_m^\alpha f(x) = \frac{m}{\Gamma(\alpha)} \int_x^\infty (x^m - u^m)^{\alpha-1} u^{m-1} f(u)\, du. \tag{A.2}$$

We can interpret this as a Stieljes integral where we are integrating with respect to the function x^m. But further, it is useful to consider power weights of these operators. Let us define

$$I_{m,}^{\eta,\alpha} f(x) = x^{-\eta-\alpha} I_m^\alpha x^\eta f(x) \tag{A.3}$$

$$K_m^{\eta,\alpha} f(x) = x^{-\eta-\alpha} K_m^\alpha x^\eta f(x) \tag{A.4}$$

which are called *Erdélyi-Kober operators* after the two mathematicians who popularized their use. Although we have not specified the constraints on the orders,

it is taken by convention that m is a positive integer, and $\alpha, \eta > 0$. Obviously the case when we have $\eta = 0$ and $m = 1$ reduces to the case which we studied. The case when $m = 2$ has deep connections with the Hankel transform (McBride [27]). The Hankel transform is used in polar and spherical coordinates, thus it not by coincidence that Section 5.2 is solved by these operators. The operators satisfy the semi-group property

$$K_m^{\eta,\alpha} K_m^{\eta+\alpha,\beta} \varphi = K_m^{\eta,\alpha+\beta} \varphi. \tag{A.5}$$

(aa)

(bb)

(cc)

Figure A.1: Arthur Erdélyi (1908 - 1977), Hermann Kober (1888 - 1973) and Adam McBride (1946 -)

The problem with trying to use convolutions when defining Erdélyi-Kober operators on distributions lies with the fact that point-wise multiplication of distributions is ill-defined. Because of this, we observe the integration by parts formula for fractional calculus (2.9), so that

$$(I^\lambda f, \varphi) = (f, K^\lambda \varphi). \tag{A.6}$$

This definition makes sense if the space of test functions is invariant with respect to fractional integration is invariant. We start with the simplest case, where Ω is a finite interval. The space of smooth functions vanishing at b is ideal

$$X([a,b]) = C_b^\infty([a,b]) := \{\varphi \in C^\infty([a,b]),\ \varphi^{(k)}(b) = 0 \text{ for } k = 0, 1, 2 \ldots\}. \tag{A.7}$$

However, the definition $X'([a,b])$ is a meaningless one as our definition of a dual space requires $[a,b]$ to be open. To fix this, we extend the test function to the positive reals by defining the function to be zero outside the interval. We follow

[12] and construct a fundamental space over a finite interval, then the union will cover the space of test functions with compact support.

Let $l > 0$. We consider a test function space on $[0, l]$ as above. Let $\mathcal{I}_l$ be the set of complex valued smooth functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that $\operatorname{supp}\varphi \subset [0, l]$ and for which the

$$\|\varphi\|_k := \sup_{x>0}\{x^k|\varphi^{(k)}(x)|\} \tag{A.8}$$

is finite. This is a Fréchet space (built up using countable many seminorms) on $\mathcal{I}_l$ and in particular $\|\cdot\|_0$ is a norm. We define the set $\mathcal{I} = \bigcup_{l=1}^{\infty}\mathcal{I}_l$.

We define on $\mathcal{I}$ the Erdélyi-Kober operator

$$K_m^{\eta,\alpha}f(x) := \frac{mx^{m\eta}}{\Gamma(\alpha)}\int_x^\infty (u^m - x^m)^{\alpha-1}\varphi(u)\frac{u^{m-1}}{u^{(\alpha+\eta)m}}du \tag{A.9}$$

$$= \frac{m}{\Gamma(\alpha)}\int_1^\infty (t^m - 1)^{\alpha-1}\varphi(xt)\frac{t^{m-1}}{t^{-\alpha m-\eta m}}dt \tag{A.10}$$

If we differentiate A.10 with respect to x which we can do if $x > 0$

$$\frac{d^k}{dx^k}(K_m^{\eta,\alpha}\varphi(x)) = \frac{m}{\Gamma(\alpha)}\int_1^\infty (t^m - 1)^{\alpha-1}\varphi^{(k)}(xt)\frac{t^{m-1}}{t^{-\alpha m-\eta m}}dt \tag{A.11}$$

so that $\|K_m^{\eta,\alpha}\varphi\|_k$ is finite and so $K_m^{\eta,\alpha}$ is a continuous map from $\mathcal{I}$ to itself. However, it turns out the most natural function space to study these operators on is what is now called McBride's space. The interested reader is invited to see [27].

A.2 Kiryakova's generalized fractional calculus

Because of the popularity of the Erdélyi-Kober operators, there have been attempts to generalize even these operators by changing the integral kernel. The first attempt in this approach was by Saigo [37] who defined, using the hypergeometric

function,

$$I^{\alpha,\beta,\eta} f(x) := \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)\, dt \tag{A.12}$$

$$J^{\alpha,\beta,\eta} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)\, dt. \tag{A.13}$$

For the properties of this operator see Saigo and Ryabogin [38] and the references therein.

This concept was pushed furthest by Kiryakova [21] who used the Meijer G-function to define a very flexible fractional calculus.

Figure A.2: Virginia Kiryakova

Definition A.2.1. Let $m \geq 1$ be an integer, $\beta > 0, \gamma_1, \ldots, \gamma_m$ and $\delta_1 > 0, \ldots, \delta_m > 0$ be arbitrary real numbers. Define $(\gamma_1, \ldots, \gamma_m)$ and $\delta = (\delta_1, \ldots, \delta_m)$ as the *multiweight* and *multiorder* of the integration respectively. We define the *multiple Erdélyi-Kober operators* as

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) := \int_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+\delta_k)_1^m \\ (\delta_k)_1^m \end{matrix}\right] f(x\sigma^{1/\beta})\, d\sigma \tag{A.14}$$

$$W_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) := \int_1^\infty G_{m,m}^{m,0}\left[\frac{1}{\sigma} \,\middle|\, \begin{matrix} (\gamma_k+\delta_k+1)_1^m \\ (\gamma_k+1)_1^m \end{matrix}\right] f(x\sigma^{1/\beta})\, d\sigma \tag{A.15}$$

and then each operator of the forms

$$Rf(x) = x^{\beta\delta_0} I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) \tag{A.16}$$

$$Wf(x) = x^{\beta\delta_0} W_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) \text{ with } \delta_0 \geq 0 \tag{A.17}$$

are said to be *generalized R.-L. fractional integral* and *generalized Weyl fractional integrals* respectively.

Here, the choice of kernel function is a subclass of the Meijer's G-function where

$$G_{m,m}^{m,0}\left[\sigma \left| \begin{array}{c} (a_k)_1^m \\ (b_k)_1^m \end{array} \right. \right] = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{\Gamma(b_1 - s)\cdot\Gamma(b_2 - s)\ldots\Gamma(b_m - s)}{\Gamma(a_1 - s)\cdot\Gamma(a_2 - s)\ldots\Gamma(a_k - s)} \sigma^s \, ds \qquad \text{(A.18)}$$

and $\mathcal{L}$ is suitably chosen contour. For the properties of the G-function see Erdlyi et al. [10]. The restriction of the order of the G-function to $(m, 0, m, m)$ is due to the nice properties this class of functions, and hence, the operators also have very nice properties.

1. (Convolution Theorem) The operator $I_{\beta,m}^{(\gamma_k),(\delta_k)}$ can be represented as a Mellin convolution

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x^{1/\beta}) = G_{m,m}^{m,0}\left[\sigma \left| \begin{array}{c} (\gamma_k + \delta_k)_1^m \\ (\delta_k)_1^m \end{array} \right. \right] \circ f(x^{1/\beta}).$$

2. (Decomposition Theorem) Each multiple Erdélyi-Kober operator can be represented by a m-tuple composition of commuting classical Erdélyi-Kober fractional integrals:

$$I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) = \left(I_\beta^{\gamma_1,\delta_1} \circ \ldots \circ I_\beta^{\gamma_m,\delta_m}\right) f(x).$$

   The converse gives the semigroup property.

3. (Commutation Theorem) The multiple Erdélyi-Kober operators commute with each other. This is because the G-function is *symmetric* in the parameters $(\gamma_k + \delta_k)_1^m$ and $(\gamma_k)_1^m$.

For proofs of these results, and a discussion on how to extend the operators to incorporate derivatives, see Kiryakova [21].

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